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Analysis and Connectedness of Four Dimensional Designs

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Summary

An attempt has been made to derive the F matrix of the reduced normal equations in the most general set-up of four dimensional designs. Some theorems pertaining to connectedness have been established.

Key Words: F - Matrix, Normal Equations, Four Dimensional Designs, Commutative Design, Connectedness, Orthogonality.

Introduction

Pearce [6] has considered designs of the type 0:00: SSS. Such designs find use when further set of treatments are added to a Latin square design. He also has given an analysis of such designs. Clarke [3] has considered the analysis of a particular type of designs eliminating three - way heterogeneity. The construction problem of such four dimensional designs is tackled by pathoff [7], Bose *et al.* [2], Agrawal *et al.* [1], Pal *et al.* [4]. Pal and Katyal [5] have considered designs having multi - way heterogeneity. The F-matrix of the reduced normal equations in the analysis of four - factor design is derived. The results giving treatment connectedness of such designs in four dimensional case is also discussed.

2. *Model Under Three-Way Elimination of Heterogeneity Set-up*

Consider a four way design having v treatments arranged in b rows, b′ columns and b″ symbols. Let Y_{ijhlk} denote the yield corresponding to the k-th observation in the j-th row, h-th column, l-th symbol having i-th treatment.

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The linear additive model under the above four-way design set-up is:

$$Y_{ijhlk} = \mu + \alpha_i + \beta_j + \gamma_h + \theta_l + E_{ijhlk}$$

where μ : the general effect,

α_i : effect of i - th treatment; $i = 1, 2, \ldots, v$,

β_j : effect of j - th row ; $j = 1, 2, \ldots, b$,

γ_h : effect of h - th column; $h = 1, 2, \ldots, b'$,

θ_l : effect of l - th symbol; $l = 1, 2, \ldots, b''$,

and E_{ijhlk}'s are error components which are i.i.d..N $(0, \sigma^2)$, k can assume at the most m-1 values, if we are considering this four - way design as an m - ary design.

Let M_{vxb}, $N_{vxb'}$, $P_{vxb''}$ be the treatment vs row, the treatment vs column, the treatment vs symbol incidence matrices respectively. And $R_{bxb'}$, $S_{bxb''}$, $U_{b'xb''}$ be the row vs column, row vs symbol and column vs symbol incidence matrices respectively. Further, let

$\underline{T} = (T_1, T_2, \ldots, T_v)'$; $\underline{B} = (B_1, B_2, \ldots, B_b)'$;

$\underline{C} = (C_1, C_2, \ldots, C_{b'})'$; $\underline{D} = (D_1, D_2, \ldots, D_{b''})'$;

$\underline{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_v)'$; $\underline{\beta} = (\beta_1, \beta_2, \ldots, \beta_b)'$;

$\underline{\gamma} = (\gamma_1, \gamma_2, \ldots, \gamma_{b'})'$; $\underline{\theta} = (\theta_1, \theta_2, \ldots, \theta_{b''})'$;

$\underline{r} = (r_1, r_2, \ldots, r_v)$ and $r^\delta = \text{diag}(r_1, r_2, \ldots, r_v)$

$\underline{k} = (k_1, k_2, \ldots, k_b)$ and $k^\delta = \text{diag}(k_1, k_2, \ldots, k_b)$;

$\underline{e} = (e_1, e_2, \ldots, e_{b'})$ and $e^\delta = \text{diag}(e_1, e_2, \ldots, e_{b'})$

$\underline{f} = (f_1, f_2, \ldots, f_{b''})$ and $f^\delta = \text{diag}(f_1, f_2, \ldots, f_{b''})$

and n, the total number of experimental units.

Let $\underline{1}$ denote a column vector with all elements unity, J the matrix of all ones of appropriate order,

and I the identity matrix.

Then $M'\underline{1} = \underline{k}$, $N'\underline{1} = \underline{e}$, $P'\underline{1} = \underline{f}$, $M\underline{1} = N\underline{1} = P\underline{1} = \underline{r}$.

The normal equations are:

$$\underline{T} = \underline{r}\,\hat{\mu} + r^{\delta}\,\hat{\underline{\alpha}} + M\,\hat{\underline{\beta}} + N\,\hat{\underline{\gamma}} + P\,\hat{\underline{\theta}} \tag{2.1}$$

$$\underline{B} = \underline{k}\,\hat{\mu} + M'\,\hat{\underline{\alpha}} + K^{\delta}\,\hat{\underline{\beta}} + R\,\hat{\underline{\gamma}} + S\,\hat{\underline{\theta}} \tag{2.2}$$

$$\underline{C} = \underline{e}\,\hat{\mu} + N'\,\hat{\underline{\alpha}} + R'\,\hat{\underline{\beta}} + e^{\delta}\,\hat{\underline{\gamma}} + U\,\hat{\underline{\theta}} \tag{2.3}$$

$$\underline{D} = \underline{f}\,\hat{\mu} + P'\hat{\underline{\alpha}} + S'\,\hat{\underline{\beta}} + U'\,\hat{\underline{\gamma}} + f^{\delta}\,\hat{\underline{\theta}} \tag{2.4}$$

After algebraic manipulation we get the reduced normal equation to estimate treatment effect as,

$$F\,\hat{\underline{\alpha}} = \underline{Q} \tag{2.5}$$

Where,

$$\begin{aligned} F = & \Big[(r^{\delta} - Mk^{-\delta}M') - (P - Mk^{-\delta}S)\,(f^{\delta} - S'k^{-\delta}S)^{G}\,(P' - S'k^{-\delta}M')\Big] \\ & -\Big[\big\{(N - Mk^{-\delta}R) - (P - Mk^{-\delta}S)(f^{\delta} - S'k^{-\delta}S)^{G}\,(U' - S'k^{-\delta}R)\big\} \\ & \big\{(e^{\delta} - R'k^{-\delta}R) - (U - R'k^{-\delta}S)\,(f^{\delta} - S'k^{-\delta}S)^{G}\,(U' - S'k^{-\delta}R)\big\}^{G} \\ & \big\{(N' - R'k^{-\delta}M') - (U' - R'k^{-\delta}S)(f^{\delta} - S'k^{-\delta}S)^{G}\,(P' - S'k^{-\delta}M')\big\}\Big] \end{aligned} \tag{2.5a}$$

and

$$\begin{aligned} \underline{Q} = & \Big[(\underline{T} - Mk^{-\delta}\underline{B}) - (P - Mk^{-\delta}S)\,(f^{\delta} - S'k^{-\delta}S)^{G}\,(\underline{D} - S'k^{-\delta}\underline{B})\Big] \\ & -\Big[\big\{(N - Mk^{-\delta}R) - (P - Mk^{-\delta}S)\,(f^{\delta} - S'k^{-\delta}S)^{G}\,(U' - S'k^{-\delta}R)\big\} \\ & \big\{(e^{\delta} - R'k^{-\delta}R) - (U - S'k^{-\delta}S)\,(f^{\delta} - S'k^{-\delta}S)^{G}\,(U' - S'k^{-\delta}R)\big\}^{G} \end{aligned}$$

$$\left\{ (\underline{C} - R' \; k^{-\delta} \underline{B}) - (U - R' \; k^{-\delta} S) \, (f^{\delta} - S' \, k^{-\delta} S)^{G} \, (\underline{D} - S' \, k^{-\delta} \underline{B}) \right\} \Big] \tag{2.5b}$$

Let,
$$F + \frac{1}{n} (\underline{r} \, \underline{r}') = r^{\delta} \; (I - M_o) \tag{2.6}$$

$$I - M_o = r^{-\delta} F + \frac{1}{n} (\underline{1} \, \underline{r}')$$

$$\therefore \quad M_o = I - r^{-\delta} F - \frac{1}{n} (\underline{1} \, \underline{r}') \tag{2.7}$$

3. *Generalised Row Column Designs*

DEFINITION 3.1: A generalised row - column design of o-th order is defined as a design in which the estimates of row effects ignoring treatment effects are orthogonal to the estimates of column effects ignoring treatment effects.

DEFINITION 3.2: A generalised row - column design of first order is a design in which

(a) the estimates of row effects are orthogonal to the estimates of column effects,

(b) the estimates of row effects are orthogonal to the estimates of symbol effects and

(c) the estimates of column effects are orthogonal to estimates of symbol effects.

The other ways of classifications in conditions (a), (b) & (c) above are ignored in each case.

If we write the matrix equations parallel to (2.5) for estimating $\hat{\underline{\beta}}$, $\hat{\underline{\gamma}}$ and $\hat{\underline{\theta}}$ i.e. we deal with equations of the type:

$$F_1 \, \hat{\underline{\beta}} = \underline{Q}_1 \, , \; F_2 \, \hat{\underline{\gamma}} = \underline{Q}_2 \, , \quad F_3 \, \hat{\underline{\theta}} = \underline{Q}_3$$

The equations of the type (2.6) are

$$F_1 + \frac{1}{n} (\underline{k} \, \underline{k}') = k^{\delta} (I - M_{01}) \; ,$$

$$F_2 + \frac{1}{n}(\underline{e}\,\underline{e}') = e^{\delta}(I - M_{02}),$$

$$F_3 + \frac{1}{n}(\underline{f}\,\underline{f}') = f^{\delta}(I - M_{03}),$$

where

$$M_{01} = I - k^{-\delta} F_1 - \frac{1}{n}(\underline{1}\,\underline{k}'),$$

$$M_{02} = I - e^{-\delta} F_2 - \frac{1}{n}(\underline{1}\,\underline{e}')$$

and $$M_{03} = I - f^{-\delta} F_3 - \frac{1}{n}(\underline{1}\,\underline{f}')$$

DEFINITION 3.3: A generalised row - column design of first order is said to be commutative, if,

$$M_{0i} M_{0j} = M_{0j} \quad (\text{for } i \neq j; i,j = 1,2,3)$$

One can note that commutativity of these M_0 - matrices implies occurrences of the same set of eigen vectors for M_{01}, M_{02} & M_{03}.

DEFINITION 3.4: A generalised row-column design of first order is said to be orthogonal if,

$$\text{Cov}(\hat{\underline{\beta}}\,\hat{\underline{\gamma}}) = \text{Cov}(\hat{\underline{\beta}}, \hat{\underline{\theta}}) = \text{Cov}(\hat{\underline{\gamma}}, \hat{\underline{\theta}}) = 0$$

DEFINITION 3.5: A generalised row - column design of first order is said to be treatment connected when all independent treatment contrasts are estimable in the complete four - way design.

THEOREM 3.1: The conditions

$$R = \frac{1}{n}(\underline{k}\,\underline{e}'),$$

$$S = \frac{1}{n}(\underline{k}\,\underline{f}') \quad \text{and} \quad U = \frac{1}{n}(\underline{e}\,\underline{f}')$$

are sufficient for obtaining a generalised row-column design of first order.

Proof: Recalling the definition 3.2 let us consider the condition (a), namely, the estimates of row effects are orthogonal to the estimates of column effects ignoring other classifications (i.e. treatment and symbol).

Now the reduced normal equations for estimation of row effects ($\hat{\underline{\beta}}$) and column effects ($\hat{\underline{\gamma}}$) can be written as,

$$\left.\begin{aligned} F_1 \hat{\underline{\beta}} &= \underline{Q}_1 \\ \& \quad F_1 \hat{\underline{\gamma}} &= \underline{Q}_2 \end{aligned}\right\}$$, the remaining classification's being ignored,

where F_1, F_2, Q_1, Q_2 are given by,

$$F_1 = k^{\delta} - R\,e^{-\delta}\,R'\ ,\ F_2 = e^{\delta} - R'\,k^{-\delta}\,R$$

$$\underline{Q}_1 = \underline{B} - R\,e^{-\delta}\,\underline{C}\ ,\ \underline{Q}_2 = \underline{C} - R'\,k^{-\delta}\,\underline{B}$$

In fact, $\text{Cov}(Q_1, Q_2) = \text{Cov}(\underline{B} - Re^{-\delta}\underline{C}, \underline{C} - R'k^{-\delta}\underline{B})$

$= \text{Cov}(\underline{B}, \underline{C}) - \text{Cov}(\underline{B}, R'k^{-\delta}\underline{B}) - \text{Cov}(R\,e^{-\delta}\underline{C}, \underline{C}) + \text{Cov}(Re^{-\delta}\underline{C}, R'\,k^{-\delta}\underline{B})$

$= R\,\sigma^2 - k^{\delta}\,k^{-\delta}\,R\sigma^2 - R\,e^{-\delta}\,e^{\delta}\,\sigma^2 + R\,e^{-\delta}\,R'\,k^{-\delta}\,R\,\sigma^2$

By virtue of condition (a) (Definition 3.2) mentioned above,

$$\text{Cov}(\underline{Q}_1, \underline{Q}_2) = 0$$

Or $\quad R\,e^{-\delta}\,R'\,k^{-\delta}\,R = R$

i.e. $\quad R^{g} = e^{-\delta}\,R'\,k^{-\delta}$

$\Rightarrow \quad R = \dfrac{1}{n}(\underline{k}\,\underline{e}')$

Similarly, Starting from the reduced normal equations for row effect ($\underline{\beta}$) and symbol effects ($\underline{\theta}$) and solving as before, we get

$$S = \frac{1}{n}(\underline{k}\,\underline{f}')$$

and column effect ($\underline{\gamma}$) and symbol effects ($\underline{\theta}$) would give,

$$U = \frac{1}{n}(\underline{e}\,\underline{f}')$$

THEOREM 3.2: A commutative generalised row - column design of first order which is

(a) row treatment connected,

(b) column treatment connected,

and (c) symbol treatment connected,

may or may not be treatment connected. In fact, it could be treatment disconnected.

Proof: For a commutative generalised row-column design of first order, if a contrast of the effects of the different levels of the treatment factor, namely, $s'\underline{\alpha}$ with $s'\underline{r} = 0$ and $M_0 \underline{s} = \mu_0 \underline{s}$, where μ_0 is the efficiency factor, exists and the design is further row treatment connected, column - treatment connected and symbol-treatment connected, given by (a), (b) and (c) above, then we have

$$M_{01} \underline{s} = \mu_1 \underline{s}, \quad 0 \le \mu_1 < 1 \tag{3.6}$$

$$M_{02} \underline{s} = \mu_2 \underline{s}, \quad 0 \le \mu_2 < 1 \tag{3.7}$$

$$M_{03} \underline{s} = \mu_3 \underline{s}, \quad 0 \le \mu_3 < 1 \tag{3.8}$$

Where

μ_1 is the row - treatment efficiency factor,

μ_2 is the column - treatment efficiency factor,

μ_3 is symbol - treatment efficiency factor,

Summing (3.6), (3.7) & (3.8) we have

$$(M_{01} + M_{02} + M_{03})\underline{s} = (\mu_1 + \mu_2 + \mu_3)\underline{s}$$

Now it is not necessary that $\mu_0 = \mu_1 + \mu_2 + \mu_3$ is always less than unity. If μ_0 is less than unity, the commutative generalised row-column design of first order is also treatment connected when (a), (b) and (c) are true. Otherwise, even in case μ_0 is unity, the design becomes treatment-disconnected.

THEOREM 3.3: The conditions

$$R = M' \, r^{-\delta} N \tag{3.9}$$

$$S = M' \, r^{-\delta} P \tag{3.10}$$

& $$U = N' \, r^{-\delta} P \tag{3.11}$$

are sufficient for a generalised row-column design of first to be orthogonal.

Proof: Premultiplying both sides of (2.1) by $M' r^{-\delta}$ and substracting from (2.2), we get

$$\underline{B} - M' r^{-\delta} \underline{T} = (k^{\delta} - M' r^{-\delta} M) \hat{\underline{\beta}} + (R - M' r^{-\delta} N) \hat{\underline{\gamma}} + (S - M' r^{-\delta} P) \hat{\underline{\theta}} \tag{3.12}$$

Now premultiplying both sides of (2.1) by $N' r^{-\delta}$ and subtracting from (2.3), we get

$$\underline{C} - N' r^{-\delta} \underline{T} = (R' - N' r^{-\delta} M) \hat{\underline{\beta}} + (e^{\delta} - N' r^{-\delta} N) \hat{\underline{\gamma}} + (U - N' r^{-\delta} P) \hat{\underline{\theta}} \tag{3.13}$$

Now premultiplying both sides of (2.1) by $P' r^{-\delta}$ and substracting from (2.4), we get

$$\underline{D} - P' r^{-\delta} \underline{T} = (S' - P' r^{-\delta} M) \hat{\underline{\beta}} + (U' - P' r^{-\delta} N) \hat{\underline{\gamma}} + (f^{\delta} - P' r^{-\delta} P) \hat{\underline{\theta}} \tag{3.14}$$

Now applying the given conditions (3.9), (3.10) and (3.11) to (3.12), (3.13) and (3.14), we have

$$\underline{B} - M' r^{-\delta} \underline{T} = (k^{\delta} - M' r^{-\delta} M) \hat{\underline{\beta}}$$

$$\underline{C} - N' r^{-\delta} \underline{T} = (e^{\delta} - N' r^{-\delta} N) \hat{\underline{\gamma}}$$

$$\underline{D} - P' r^{-\delta} \underline{T} = (f^{\delta} - P' r^{-\delta} P) \hat{\underline{\theta}}$$

Now $\quad \text{Cov}(\underline{B} - M' r^{-\delta} \underline{T}, \; \underline{C} - N' r^{-\delta} \underline{T})$

$$= \text{Cov}(\underline{B}, \underline{C}) - \text{Cov}(M' r^{-\delta} \underline{T}, \underline{C}) - \text{Cov}(\underline{B}, N' r^{-\delta} \underline{T}) + \text{Cov}(M' r^{-\delta} \underline{T}, N' r^{-\delta} \underline{T})$$

$$= R\sigma^2 - M'r^{-\delta}\text{Cov}(\underline{T}, \underline{C}) - \text{Cov}(\underline{B}, \underline{T})\, r^{-\delta}N + M'\, r^{-\delta}\text{Cov}(\underline{T}, \underline{T})r^{-\delta}\, N$$

$$= R\sigma^2 - M\, r^{-\delta}\, N\sigma^2 - M'\, r^{-\delta}\, N\sigma^2 + M'\, r^{-\delta}\, r^{\delta}\, r^{-\delta}\, N\, \sigma^2$$

$$= (R - M'\, r^{-\delta}\, N)\, \sigma^2 = 0$$

Therefore, $\text{Cov}(\hat{\underline{\beta}}, \hat{\underline{\gamma}}) = 0$.

Similarly, we can show that

$$\text{Cov}(\underline{B} - M'\, r^{-\delta}\, \underline{T}, \underline{D} - P'\, r^{-\delta}\, \underline{T}) = (S - M'\, r^{-\delta}\, P)\, \sigma^2$$

$$\text{Cov}(\hat{\underline{\beta}}, \hat{\underline{\theta}}) = 0$$

and

$$\text{Cov}(\underline{C} - N'\, r^{-\delta}\, \underline{T}, \underline{D} - P'\, r^{-\delta}\, \underline{T}) = (U - N'\, r^{-\delta}\, P)\, \sigma^2$$

$$\text{Cov}(\hat{\underline{\gamma}}, \hat{\underline{\theta}}) = 0$$

Hence the theorem is proved.

THEOREM 3.4: An orthogonal generalised row-column design of first order in always commutative.

Proof: A generalised row - column design of first order is orthogonal when

(i) $R = M'\, r^{-\delta}\, N$, (ii) $S = M'\, r^{-\delta}\, P$, (iii) $U = N'\, r^{-\delta}\, P$ and further from theorem 3.1,

(a) $R = \frac{1}{n}(\underline{k}\,\underline{e}')$, (b) $S = \frac{1}{n}(\underline{k}\,\underline{f}')$, (c) $U = \frac{1}{n}(\underline{e}\,\underline{f}')$

Therefore, corresponding F reduces to F^*, where

$$F^* = r^{\delta} - Mk^{-\delta}\, M' - N\, e^{-\delta}\, N' - P\, f^{-\delta}\, P' + \frac{2}{n}(\underline{r}\,\underline{r}')$$

Now we can write

$$M_0 = M_{01} + M_{02} + M_{03}\text{, where}$$

$$M_{01} = r^{-\delta}\, M\, k^{-\delta}\, M' - \frac{1}{n}(\underline{1}\ \underline{r}'),$$

$$M_{02} = r^{-\delta}\, N\, e^{-\delta}\, N' - \frac{1}{n}(\underline{1}\ \underline{r}'),$$

$$M_{03} = r^{-\delta} P f^{-\delta} P' - \frac{1}{n}(\underline{1}\ \underline{r}').$$

Now we have to show that M_{0i} and M_{0j} commute for $i \neq j$, $i, j = 1, 2, 3$, e.g. we show that $M_{01} M_{02} = M_{02} M_{01}$:

$$\text{L.H.S.} = \left\{ r^{-\delta} M k^{-\delta} M' - \frac{1}{n}(\underline{1}\,\underline{r}') \right\} \left\{ r^{-\delta} N e^{-\delta} N' - \frac{1}{n}(\underline{1}\,\underline{r}') \right\}$$

$$= r^{-\delta} M k^{-\delta} M' r^{-\delta} N e^{-\delta} N' - \frac{1}{n}(\underline{1}\,\underline{r}')\, r^{-\delta} N e^{-\delta} N' - \frac{1}{n} r^{-\delta} M k^{-\delta} M' (\underline{1}\,\underline{r}')$$

$$+ \frac{1}{n}(\underline{1}\,\underline{r}') \frac{1}{n}(\underline{1}\,\underline{r}')$$

$$\text{R.H.S.} = \left\{ r^{-\delta} N e^{-\delta} N' - \frac{1}{n}(\underline{1}\,\underline{r}') \right\} \left\{ r^{-\delta} M k^{-\delta} M' - \frac{1}{n}(\underline{1}\,\underline{r}') \right\}$$

$$= r^{-\delta} N e^{-\delta} N' r^{-\delta} M k^{-\delta} M' - (r^{-\delta} N e^{-\delta} N') \frac{1}{n}(\underline{1}\,\underline{r}') - \frac{1}{n}(\underline{1}\,\underline{r}')(r^{-\delta} M k^{-\delta} M')$$

$$+ \frac{1}{n}(\underline{1}\,\underline{r}') \frac{1}{n}(\underline{1}\,\underline{r}').$$

Comparing first terms on both the sides,

since $(Mk^{-\delta}M')\, r^{-\delta}\, (Ne^{-\delta}N') = (Ne^{-\delta}N')\, r^{-\delta}\, (Mk^{-\delta}M')$

and similarly for second and third terms, we see that LHS = RHS,

i.e. $M_{01} M_{02} = M_{02} M_{01}$,

and similarly we can show that $M_{01} M_{03} = M_{03} M_{01}$, and $M_{03} M_{02} = M_{02} M_{03}$, i.e. an orthogonal generalised row-column design of first order is always commutative.

THEOREM 3.5: An orthogonal generalised row - column design of first order which is row - treatment, column - treatment and symbol -treatment connected is always treatment connected.

Proof: consider an orthogonal generalised row - column design of first order.

We shall consider two exhaustive cases.

Case - 1

Let a particular treatment effect contrast be estimated with $(\mu_2 \neq 0) < 1$ efficiency (= loss of information) from the matrix M_{02}, we have

$$\left(r^{-\delta} N e^{-\delta} N' - \frac{1}{n}(\underline{1}\,\underline{r}')\right)\underline{s} = \mu_2\,\underline{s} \tag{3.15}$$

Let μ_1 and μ_3 be loss of information or efficiency, when the same treatment effect contrast is estimated from M_{01} and M_{03} respectively, i.e.

$$\left(r^{-\delta} M k^{-\delta} M' - \frac{1}{n}(\underline{1}\,\underline{r}')\right)\underline{s} = \mu_1\,\underline{s} \tag{3.16}$$

and $$\left(r^{-\delta} P e^{-\delta} P' - \frac{1}{n}(\underline{1}\,\underline{r}')\right)\underline{s} = \mu_3\,\underline{s} \tag{3.17}$$

Premultiplying both sides of (3.15) by

$$\left(r^{-\delta} M k^{-\delta} M' - \frac{1}{n}(\underline{1}\,\underline{r}')\right)$$

and applying the condition that the design is orthogonal generalised row - column design of first order, we have,

$$\left(r^{-\delta} M k^{-\delta} M' - \frac{1}{n}(\underline{1}\,\underline{r}')\right)\left(r^{-\delta} N k^{-\delta} N' - \frac{1}{n}(\underline{1}\,\underline{r}')\right)\underline{s}$$

$$= \mu_2\left(r^{-\delta} M k^{-\delta} M' - \frac{1}{n}(\underline{1}\,\underline{r}')\right)\underline{s}$$

$$\Rightarrow \left[\left(r^{-\delta} M k^{-\delta} R e^{-\delta} N' - \frac{1}{n}(\underline{1}\,\underline{e}' e^{-\delta} N')\right)\right.$$

$$\left. - \left(\frac{1}{n}\left(r^{-\delta} M k^{-\delta}\,\underline{k}\,\underline{r}'\right) - \frac{1}{n}(\underline{1}\,\underline{r}')\right)\right]\underline{s}$$

$$= \mu_2\left(r^{-\delta} M k^{-\delta} M' - \frac{1}{n}(\underline{1}\,\underline{r}')\right)\underline{s}$$

$$\Rightarrow \quad \left[\left(r^{-\delta}Mk^{-\delta}\frac{1}{n}(\underline{k}\,\underline{e}')e^{-\delta}N'\right) - \frac{1\,e'\,e^{-\delta}N'}{n} - r^{-\delta}Mk^{-\delta}\,\underline{k}\,\underline{r}'/n + \frac{1}{n}(\underline{1}\,\underline{r}')\right]\underline{s}$$

$$= \mu_2\left(r^{-\delta}M\,k^{-\delta}M' - \frac{1}{n}(\underline{1}\,\underline{r}')\right)\underline{s}$$

$$\Rightarrow \quad \left[\frac{1}{n}(\underline{1}\,\underline{r}') - \frac{1}{n}(\underline{1}\,\underline{r}') - \frac{1}{n}(\underline{1}\,\underline{r}') + \frac{1}{n}(\underline{1}\,\underline{r}')\right]\underline{s} \; (= o\underline{s})$$

$$= \mu_2\left(r^{-\delta}M\,k^{-\delta}M' - \frac{1}{n}(\underline{1}\,\underline{r}')\right)\underline{s} \qquad (3.18)$$

But R.H.S. by (3.18) is $\mu_1\,\underline{s}$.

Hence $\mu_1 = 0$ if $\mu_2 \neq 0$.

Similarly, if we multiply both sides of (3.15) by

$\left(r^{-\delta}P\,f^{-\delta}P' - \frac{1}{n}(\underline{1}\,\underline{r}')\right)$ and Proceeding as before, we can show that $\mu_3 = 0$ if $\mu_2 \neq 0$.

But for orthogonal generalised row - column design of first order

$$\mu = \mu_1 + \mu_2 + \mu_3$$

and therefore, $\mu = \mu_2 < 1$ in this case and the design is treatment connected.

Similarly, we can show that if a particular treatment effect contrast be estimated with μ_1 loss of information from the matrix M_{01}, then

$M_{01}\,S^* = \mu_1\,s^*$, then we can see that $\mu_2 = 0$, $\mu_3 = 0$ for $\mu_1 < 1$, and when treatment effect contrast be estimated with loss of information from the matrix M_{03}.

Case 2:

Let a particular treatment effect contrast be estimated with $\mu_1 = 0$, $\mu_2 = 0$ and $\mu_3 = 0$ efficiencies from the matrices M_{01}, M_{02} and M_{03} respectively, then $\mu = \mu_1 + \mu_2 + \mu_3 = 0$ that is efficiency with respect to M_0, namely $\mu_o = 0$

Hence, any treatment effect - contrast estimable from each of row, column and symbol classifications is always estimable from the whole design. Consequently, the design remains connected with respect to treatment. Hence the theorem.

Alternative Proof: For an orthogonal row-column design of first order, let

$$F_1 = r^{-\delta} - M k^{-\delta} M'$$

$$F_2 = r^{-\delta} - N e^{-\delta} N'$$

$$F_3 = r^{-\delta} - P f^{-\delta} P'$$

$$F_1 r^{-\delta} F_2 = (I - M k^{-\delta} M' r^{-\delta})(r^{\delta} - N e^{-\delta} N')$$

$$= r^{\delta} - N e^{-\delta} N' - M k^{-\delta} M' + \frac{1}{n}(\underline{r}\,\underline{r}') .$$

$$F_1 r^{-\delta} F_2 r^{-\delta} F_3 = \left(r^{\delta} - N e^{-\delta} N' - M k^{-\delta} M' + \frac{1}{n}\underline{r}\,\underline{r}'\right)(I - r^{-\delta} P f^{-\delta} P')$$

$$= r^{\delta} - N e^{-\delta} N' - M k^{-\delta} M' + \frac{1}{n}(\underline{r}\,\underline{r}') - P f^{-\delta} P'$$

$$+ N e^{-\delta} N' r^{-\delta} P f^{-\delta} P' + M k^{-\delta} M' r^{-\delta} P f^{-\delta} P' - \frac{1}{n}(\underline{r}\,\underline{r}')(r^{-\delta} P f^{-\delta} P')$$

It is easy to check that each of the last three terms on the right is equal $\frac{1}{n}(\underline{r}\,\underline{r}')$

Therefore,

$$F_1 r^{-\delta} F_2 r^{-\delta} F_3 = r^{-\delta} - M k^{-\delta} M' - N e^{-\delta} N' - P f^{-\delta} P' + \frac{2}{n}(\underline{r}\,\underline{r}')$$

$$= F^*$$

Now it can be shown that

$$F^* + \frac{1}{n}(\underline{r}\,\underline{r}') = \left(F_1 + \frac{1}{n}(\underline{r}\,\underline{r}')\right) r^{-\delta} \times \left(F_2 + \frac{1}{n}(\underline{r}\,\underline{r}')\right) r^{-\delta} \left(F_3 + \frac{1}{n}(\underline{r}\,\underline{r}')\right) r^{-\delta}$$

Thus when

Rank F_1 = Rank F_2 = Rank F_3 = $v - 1$, then

Rank $F^* = v-1$ and vice-versa.

Hence - the theorem is proved.

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